CYPRUS MATHEMATICAL SOCIETY

B΄ SELECTION COMPETITION FOR THE LYCEUM LEVEL

«Ευκλείδης»

Date: 24/02/2018 **Time duration: 10:00-14:30**

Instructions:

1. Solve all the problems showing your work.
2. Write with blue or black ink. (You may use pencil for the figures)
3. Corrector liquid (Tipp-ex) is not allowed.
4. Calculators are not permitted.

Problem 1: Consider the sequence $(\alpha_\nu)_{\nu\in\mathbb{N}}$ where $\alpha_1 = 1, \alpha_2 = 3$ and

$$\alpha_\nu = max\{a_\rho + \alpha_{\nu-\rho} \ : 1 \le \rho \le \nu - 1\} \text{ for every } \nu \ge 3$$

Show that:

α) The general term of the sequence is given by:

$$\alpha_\nu = \begin{cases} 3k & if \ \ \nu = 2k \\ 3k+1 & if \ \nu = 2k+1 \end{cases} \quad \forall k \in \mathbb{N}$$

β) $\alpha_{\nu+\mu} = \alpha_\nu + \alpha_\mu$ if and only if at least one of the indices ν, μ is even.

Problem 2: Let n be a natural number. Show that there exists a natural number m which is a multiple of n and has exactly n positive divisors.

Problem 3: The circles $c_1(O, R_1)$ and $c_2(K, R_2)$, where $R_2 > R_1$, are externally tangent at the point M. Let A be a point of c_2 which is not on the line OK, let $(\varepsilon_1), (\varepsilon_2)$ be the tangents from A on c_1, and let B, Γ be their corresponding points of intersection with c_1. The lines $MB, M\Gamma$ meet c_2 again at the points E, Z respectively. Let Λ be the point of intersection of the line EZ and the tangent of c_2 at A. Prove that $\Lambda M \perp OK$.

Problem 4: Given 2018 sets, show that there are 64 of them, say $A_1, A_2, \dots, A_{64}$, such that

$$A_i \cup A_j \ne A_k \text{ for every } i, j, k \in \{1,2,\dots,64\} \ with \ i \ne j, i \ne k, \ j \ne k$$

(I.e. the union of every two of those 64 sets is different from every other of those 64 sets.)